Nuclear Structure from Gamma-Ray Spectroscopy

2019 Postgraduate Lectures

Lecture 4: Collective Nuclear Rotation

Moment of Inertia

- Deformation provides an element of anisotropy allowing the definition of a nuclear orientation and the possibility of observing rotation

- Classically the energy associated with rotation is:

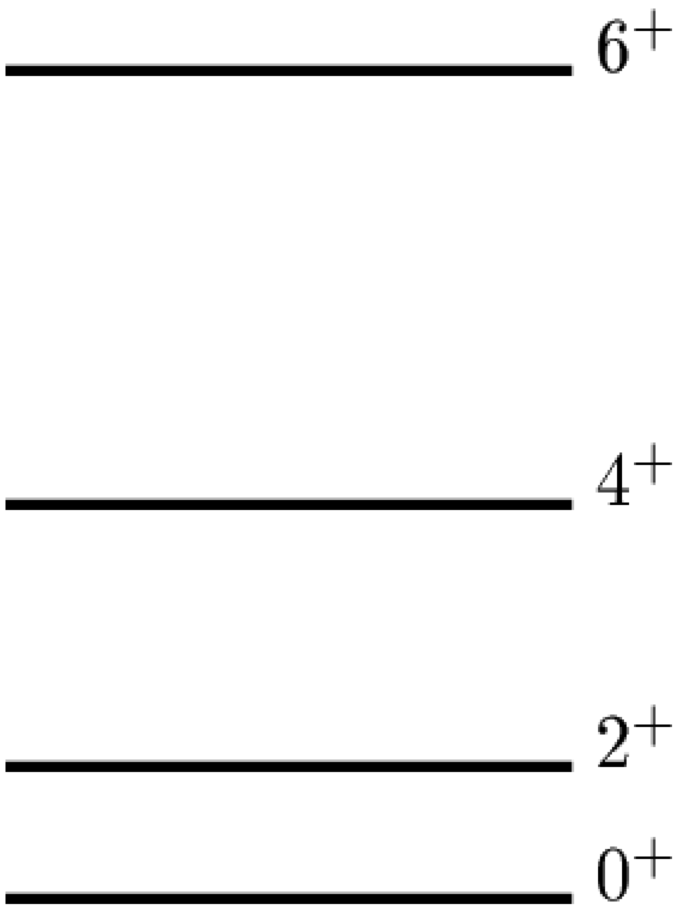
$$E_{rot} = \tfrac{1}{2}\,\Im\,\omega^2 = I^2 / 2\,\Im \;\; ; \;\; \omega = I / \Im$$

- Collective rotation involves the coherent contributions from many nucleons and gives rise to a smooth relation between energy and spin:

$$\boxed{E = (\hbar^2/2\Im)\, I[I + 1]}$$

which defines the 'static' moment of inertia, sometimes denoted $\Im^{(0)}$

Energy Levels of a Rotor



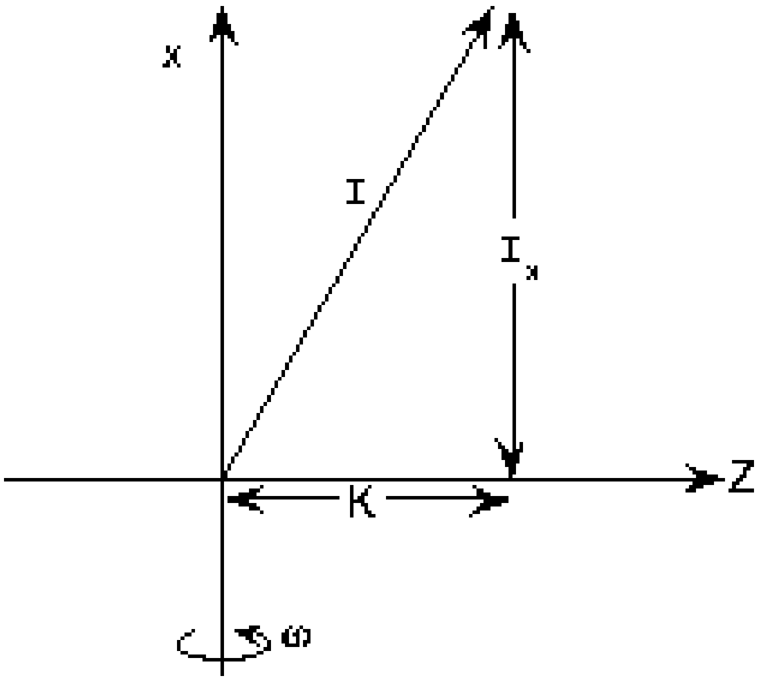
- The energy levels of a rotor are proportional to I(I+1)

- The ratios of energy levels for a rotor are:

$E(4^+)/E(2^+) = 3.333$

$E(6^+)/E(2^+) = 7.0$

Rotational Frequency



- The intensive variable ω (rotation about the x axis) is related to the extensive variable I by the relation:

$$\hbar\omega = dE/dI_x$$
$$\approx \tfrac{1}{2}[E(I+1) - E(I-1)]$$

- Here I_x is the projection of I onto the rotation axis (x):

$$\boxed{I_x = \sqrt{[I(I+1) - K^2]}\,\hbar}$$

The rotational frequency ω is distinct from the oscillator quantum ω_0. In practice $\omega \ll \omega_0$ and the collective rotation can be considered as an adiabatic motion

Rigid Body Moment of Inertia

- The rigid-body moment of inertia for a spherical nucleus is:

$$\mathfrak{I}_{rig} = (2/5)\, MR^2 = (2/5)\, A^{5/3}\, m_N\, r_0^2$$

where m_N is the mass of a nucleon ($M = A\, m_N$) and

$$R = r_0\, A^{1/3} \quad \text{with} \quad r_0 = 1.2 \text{ fm}$$

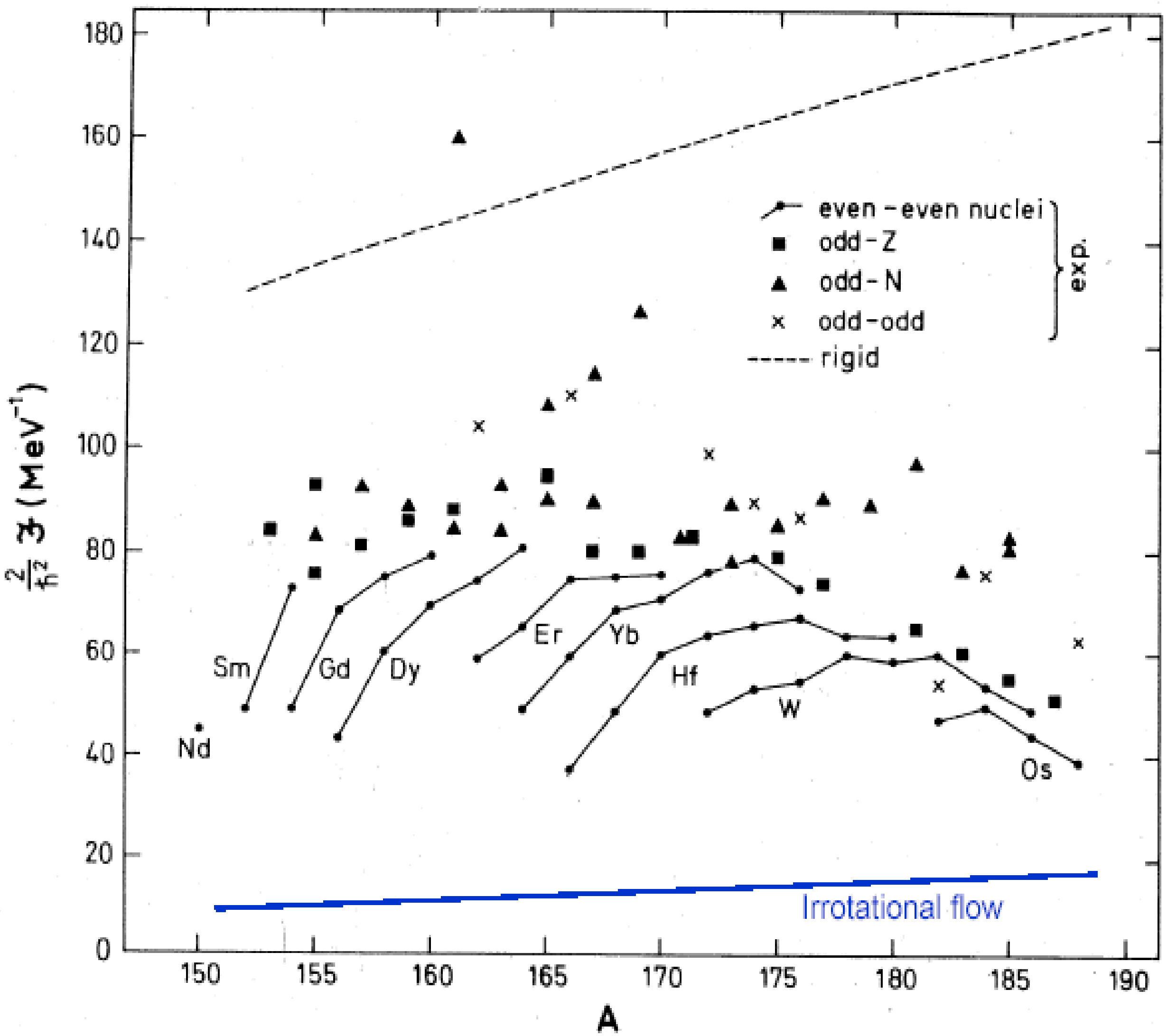
- For a deformed nucleus:

$$\mathfrak{I}_{rig} = (2/5)\, A^{5/3}\, m_N\, r_0^2\, [1 + 1/3\, \delta]$$

where $\delta = \Delta R / R_0$

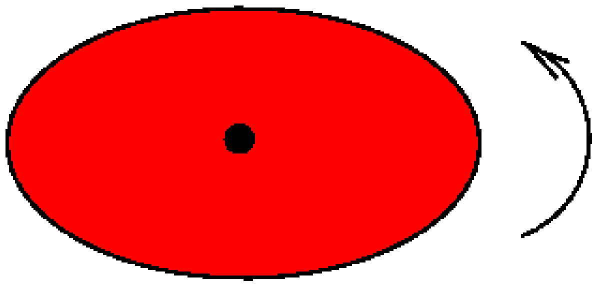
- Typically nuclear moments of inertia are less than 50% of the rigid-body value at low spin

Nuclear Moments of Inertia

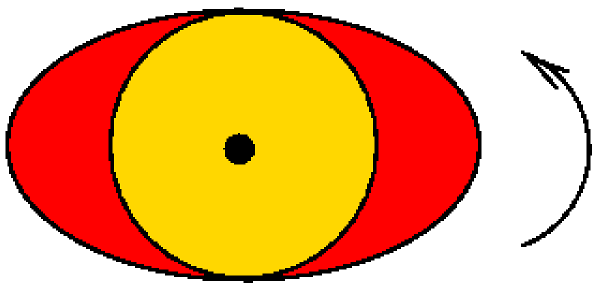


- Nuclear moments of inertia are lower than the rigid-body value – a consequence of nuclear pairing

Nuclear Rotation



Rigid body



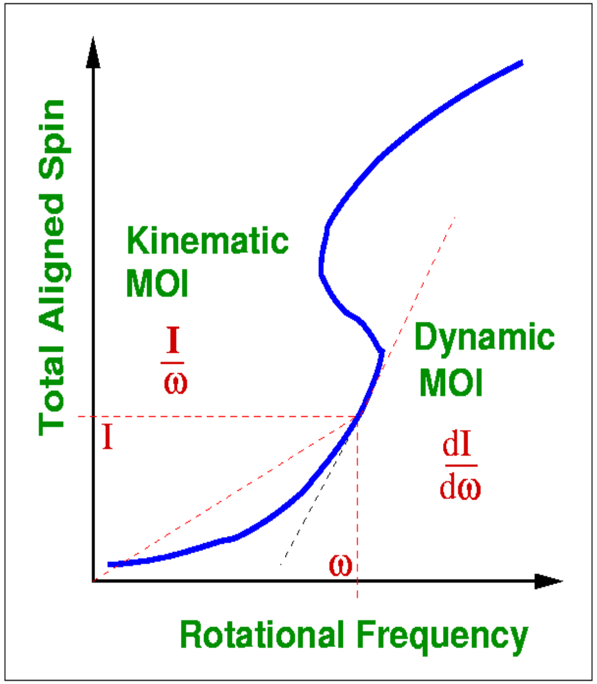
Nucleus

- The assumption of the ideal flow of an incompressible nonviscous fluid (Liquid Drop Model) leads to a hydrodynamic moment of inertia (surface waves):

$$\Im_{hydro} = \Im_{rig}\, \delta^2$$

- This estimate is much too low !
- We require short-range pairing correlations to account for the experimental values

Kinematic and Dynamic MoI's



- Assuming maximum alignment on the x-axis ($I_x \sim I$), the kinematic moment of inertia is defined:

$$\Im^{(1)} = (\hbar^2 I)\,[dE(I)/dI]^{-1} = \hbar\, I/\omega$$

- The dynamic moment of inertia (response of system to a force) is:

$$\Im^{(2)} = (\hbar^2)\,[d^2E(I)/dI^2]^{-1} = \hbar\, dI/d\omega$$

- Note that $\Im^{(2)} = \Im^{(1)} + \omega\, d\Im^{(1)}/d\omega$
- Rigid body: $\Im^{(1)} = \Im^{(2)}$ Nucleus at high spin: $\Im^{(1)} \approx \Im^{(2)}$

General Rotation

- A deformed rotor has a Hamiltonian of the form:

$$\boxed{H_{rot} = \Sigma_k A_k R_k^2}, \quad A_k = \hbar^2/2\Im_k$$

where $\Im_k$ is the moment of inertia about the k^{th} axis

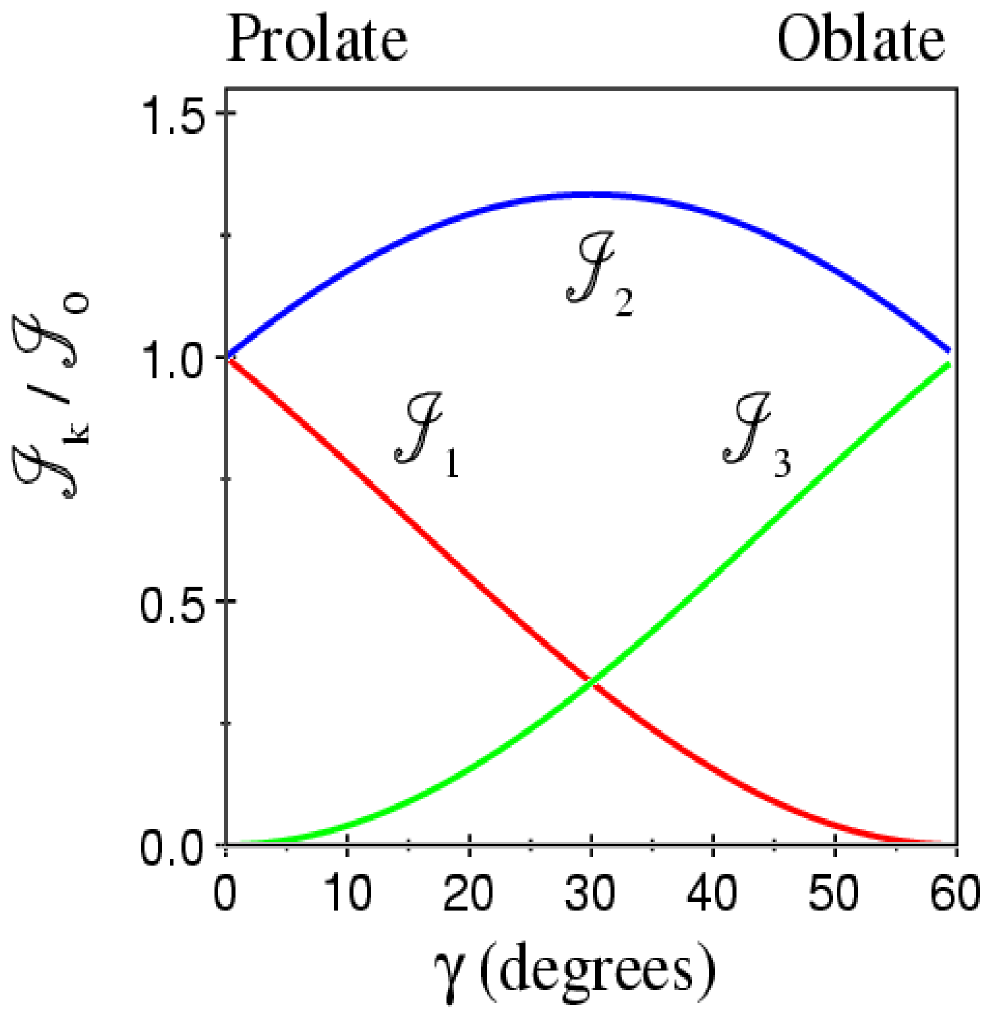
- For triaxial shapes the moments of inertia are:

$$\Im_k = (4/3)\, \Im_0 \sin^2 [\gamma + k\, 2\pi/3\ ]$$

- For an axial nucleus deformed along the z-axis, $\Im_1 = \Im_2 = \Im_0$ and $\Im_3 = 0$, and the Hamiltonian is:

$$\boxed{H_{rot} = (\hbar^2/2\Im_0)\, [R_1^2 + R_2^2] = (\hbar^2/2\Im_0)\, \underline{R}^2}$$

Irrotational Moments of Inertia



- This diagram shows the variation of the moments of inertia $\Im_k$ as a function of the triaxiality parameter γ
- For a prolate nuclear shape ($\gamma = 0°$), $\Im_1 = \Im_2$ and $\Im_3 = 0$
- For $\gamma = 30°$, $\Im_2$ reaches a maximum and this represents the 'most collective' shape

Angular Momentum Coupling

- Provided that the collective rotation is slow relative to the single-particle motion (adiabatic condition), the nuclear Hamiltonian can be separated into intrinsic and rotational parts:

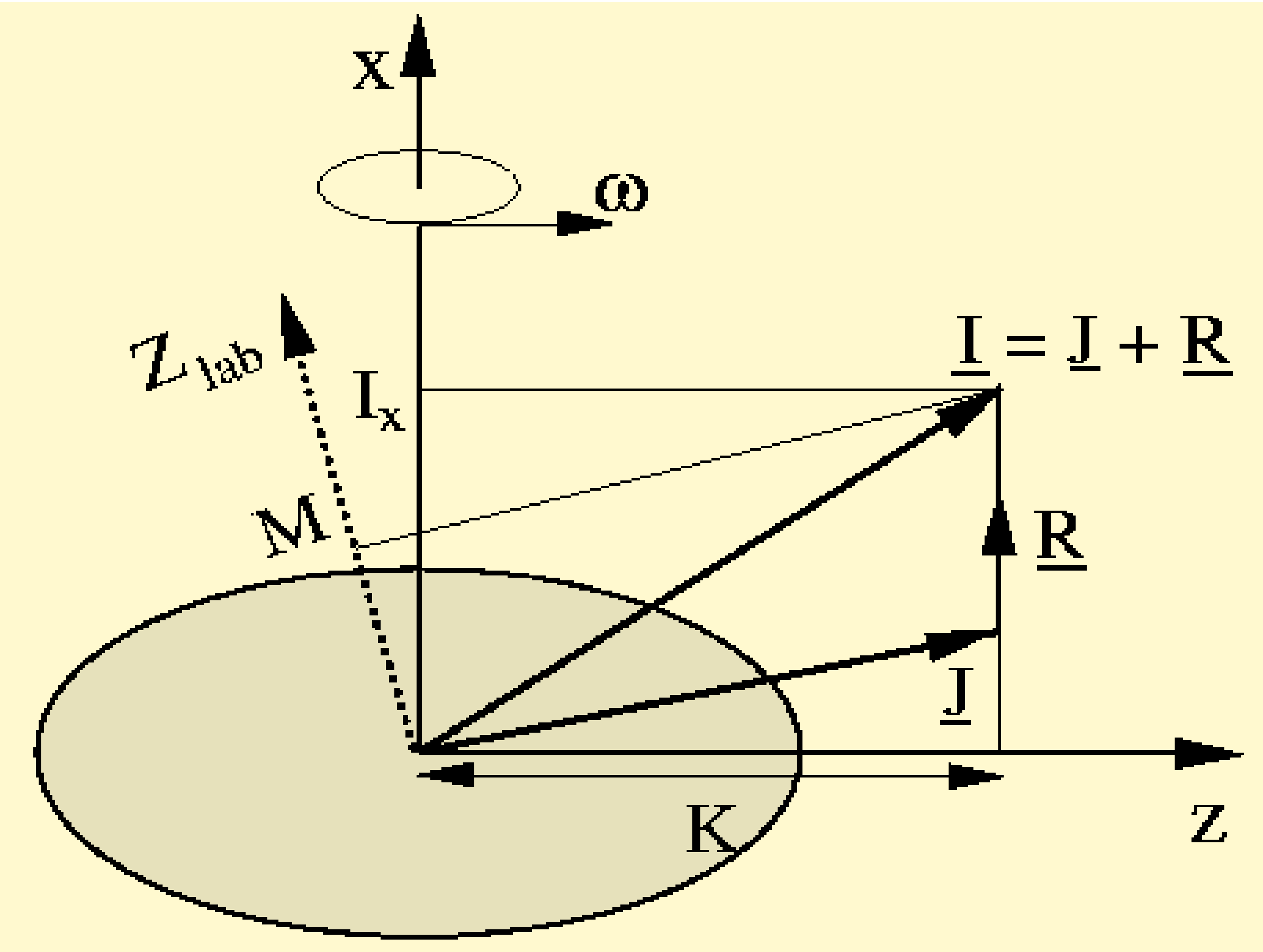
  $H = H_{int} + H_{rot}$ with eigenvalues $\Psi = \Psi_{int}\Psi_{rot}$

- The intrinsic motion has angular momentum $\underline{J}$, which is not a conserved quantity. It couples to the collective rotation $\underline{R}$ to give total spin:

  $\underline{I} = \underline{R} + \underline{J}$

- The total spin $\underline{I}$ is a <u>constant</u> of the motion together with its projection M

Various Spin Projections



Rotation Matrices

- The intrinsic wavefunction can be characterised by the K projection. The three variables I^2, M and K completely specify the state of motion The eigenfunctions are given by:

$$\boxed{\Psi_{rot} = |IMK\rangle = \sqrt{[(2I+1)/8\pi^2]}\, D_{IMK}(\theta,\varphi,\psi)}$$

where the functions D_{IMK} are '**rotation matrices**'

- Note: $\hat{I}^2 D_{IMK} = I(I+1)\hbar^2 D_{IMK}$; $\hat{I}_z D_{IMK} = K\hbar\, D_{IMK}$

$$\hat{I}_{\pm} D_{IMK} = \sqrt{[I(I+1) - K(K\mp 1)]}\,\hbar\, D_{IMK\mp 1}$$

- The rotational energy is:

$$(1/2\mathfrak{I}_x)(\hat{I}^2 - \hat{I}_z^2)\,\Psi_{rot} \quad \text{i.e.} \quad \boxed{E_{rot} = (\hbar^2/2\mathfrak{I}_x)[I(I+1) - K^2]}$$

Signature Quantum Number 'r'

- For $K = 0$, the D_{IMK} functions reduce to spherical harmonics Y_{IM} and the nuclear wavefunction is:
-

$$\Psi_{r,IMK=0} = (1/\sqrt{2})\, \Psi_{r,K=0}\, Y_{IM}$$

- The quantum number r is the 'signature', related to the invariance of the system when rotated 180° about an axis perpendicular to the symmetry axis (z): operator $R(\pi)$

- A second rotation by 180° brings the system back to its original orientation. Hence:

$$R^2(\pi)\, \Psi_{r,IMK} = r^2\, \Psi_{r,IMK} = \Psi_{r,IMK}$$

- The allowed values of r are: $(-1)^I$

Bands of Good Signature

- For $K = 0$, we may classify rotational bands in terms of the signature quantum number

- For $r = +1$, the allowed spins are:

$$I = 0, 2, 4, \ldots$$

- For $r = -1$, the allowed spins are:

$$I = 1, 3, 5, \ldots$$

- Hence for each signature we obtain a rotational band with the energy levels separated by $2\hbar$

Rotational Bands with $K \neq 0$

- For $K \neq 0$, the total nuclear wavefunction takes the antisymmetrised form in order to satisfy the rotation (reflection) symmetry:

$$\Psi_{IMK} = \sqrt{[(2I+1)/16\pi^2]}\,\{\Psi_K D_{IMK} + (-1)^{I+K}\Psi_{-K} D_{IM-K}\}$$

where Ψ_{-K} corresponds to a projection of the spin $-K$ and is obtained by the operation $R(\pi)\,\Psi_K$

- The consequence of $R(\pi)$ invariance for $K \neq 0$ is that the intrinsic states Ψ_K and Ψ_{-K}, with eigenvalues $\pm K$ of J_z, are degenerate and constitute only a single sequence of states with spins:

$$I = K, K+1, K+2, \ldots$$

i.e. states with alternating signature

Particle-Rotor Coupling

- For an axially symmetric deformed rotor:

$$H_{rot} = (\hbar^2/2\Im_0)\,\underline{R}^2 = (\hbar^2/2\Im_0)\,[\underline{I} - \underline{J}]^2$$

$$= (\hbar^2/2\Im_0)\,[\underline{I}.\underline{I} + \underline{J}.\underline{J} - 2\underline{I}.\underline{J}]$$

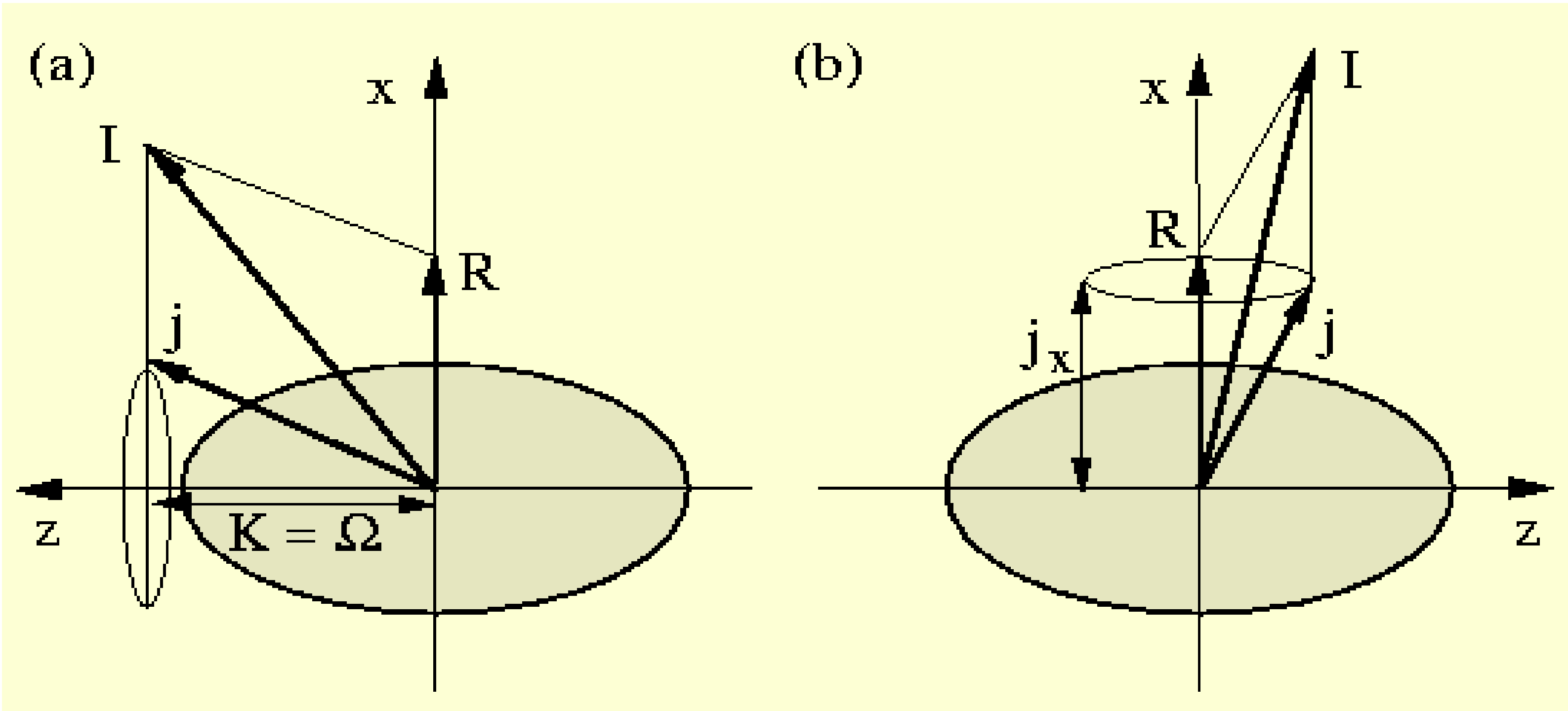
where the $\underline{I}.\underline{J}$ couples the degrees of freedom of the valence particles to the rotational motion and is analogous to the classical Coriolis and centrifugal forces

- Now consider J to consist of a single particle ($J \rightarrow j$) coupled to an even-even core

$$H_{rot} = (\hbar^2/2\Im_0)\,[(I^2 - I_z^2) + (j^2 - j_z^2) - (I_+ j_- + I_- j_+)]$$

The final term couples intrinsic and rotational states

Particle-Rotor Coupling Schemes



- (a) shows the strong-coupling limit or deformation-aligned (DAL) coupling scheme
- (b) shows the weak-coupling limit or rotation-aligned (RAL) coupling scheme

Strong Coupling (DAL)

- This limit is recognised when the level splitting of the deformed shell-model single-particle energies for different Ω values is large compared with the Coriolis perturbation, i.e. large deformation or small Coriolis matrix elements (low j, high Ω)
- The angular momentum vector $\underline{j}$ precesses around the deformation axis and K is approximately a good quantum number
- The energy spectrum is given by the set of levels:

$$\boxed{E_{rot} = (\hbar^2/2\Im_0)\,[I(I+1) - K^2]}$$

Decoupling Limit (RAL)

- For weakly deformed nuclei, or fast enough rotation, the Coriolis force may be so strong that the coupling of the valence nucleon to the deformed core is negligible

- The Coriolis force tends to align the nucleonic angular momentum $\underline{j}$ with that of the rotational angular momentum $\underline{R}$

- In this limit, the rotation band has spins:

  $I = j, j+2, j+4, \ldots$

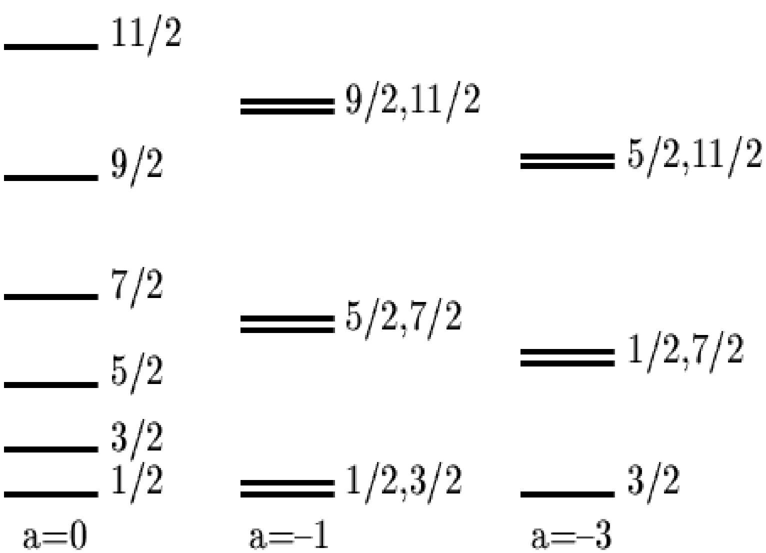
- The energies are: $\boxed{E_{rot} = (\hbar^2/2\Im_0)\,(I - j_x)\,(I - j_x + 1)}$

$K = \frac{1}{2}$ Bands in Odd-A Nuclei

- The rotational energy of a $K = \frac{1}{2}$ band is:

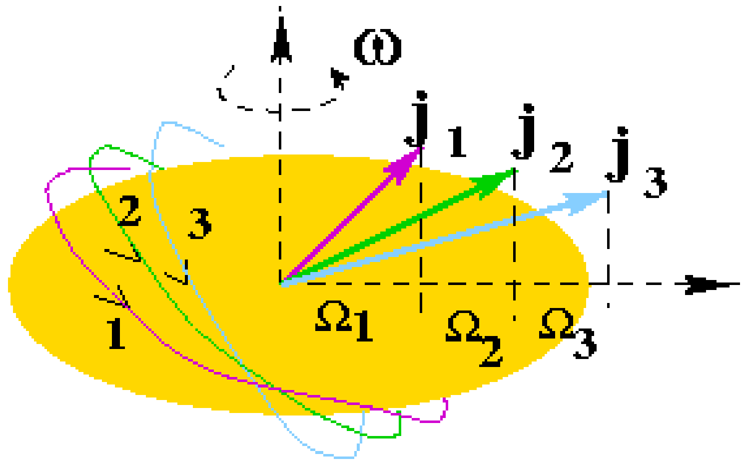
$$\boxed{E(I) = (\hbar^2/2\Im_0)\,[I(I+1) + a(-1)^{I+\frac{1}{2}}(I+\tfrac{1}{2})]}$$

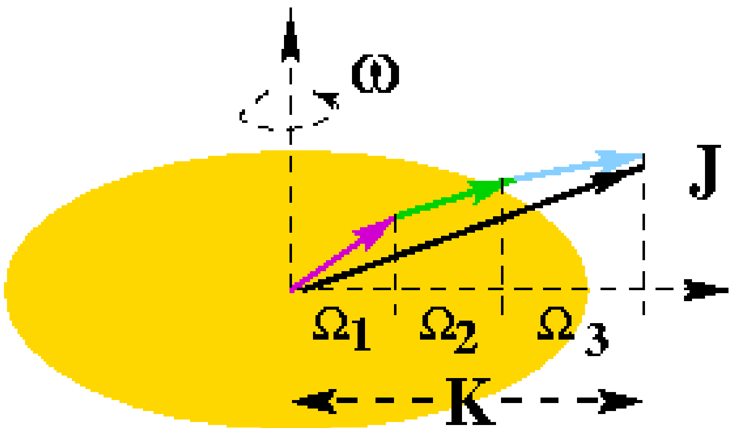
where a is the decoupling parameter



- Bands can mix if $\Delta K = \pm 1$
- For $K = \frac{1}{2}$ bands there is a diagonal matrix element of the form: $\langle K=\frac{1}{2}|j_+|K=-\frac{1}{2}\rangle$ where $j_+ = j_x + ij_y$ which perturbs the energy

High K (I_z) Bands





$K = I_z = \Sigma j_z = \Sigma\Omega$

- If we have many paired nucleons outside the closed shell in the ground state then alignment with the x-axis becomes difficult because the valence nucleons lie closer to the z-axis, i.e. they have high Ω values

- The sum K of these projections onto the deformation (z) axis is now a good quantum number

K Forbidden Transitions

- It is difficult for rotational bands with high K values to decay to bands with smaller K since the nucleus has to change the orientation of its angular momentum.

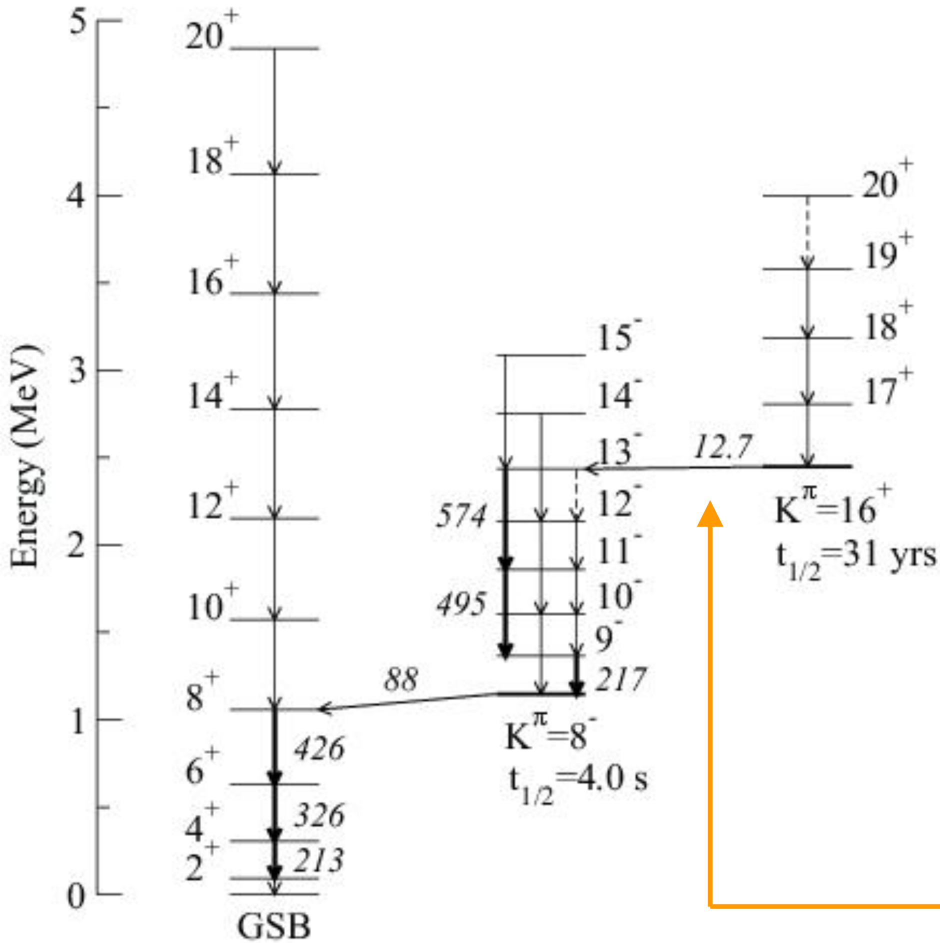
- For example, the $K^\pi = 8^-$ band head in ^{178}Hf is isomeric with a lifetime of 4 s. This is much longer than the lifetime of the rotational states built on it.

- The $K^\pi = 8^-$ band head is formed by breaking a pair of protons and placing them in the 'Nilsson configurations':

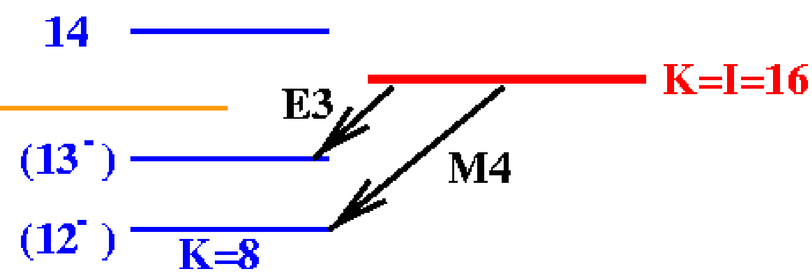
$$\Omega\ [N\ n_3\ \Lambda] = 7/2\ [4\ 0\ 4] \text{ and } 9/2\ [5\ 1\ 4]$$

- In this case: $K = 7/2 + 9/2 = 8$ and $\pi = (-1)^{N(1)}.(-1)^{N(2)} = -1$

K Isomers in ^{178}Hf



- A low lying state with spin I = 16 and K = 16 in ^{178}Hf is isomeric with a half life of 31 years !
- It is yrast (lowest state for a given spin) and is 'trapped' since it must change K by 8 units in its decay



K Forbiddenness

- Strictly, in the decay of a high-K band-head, K can only change by an amount up to the multipolarity λ of the transition

- The '<u>degree of K forbiddenness</u>' is:

  $$\nu = |\Delta K| - \lambda$$

- The '<u>hindrance factor</u>' is:

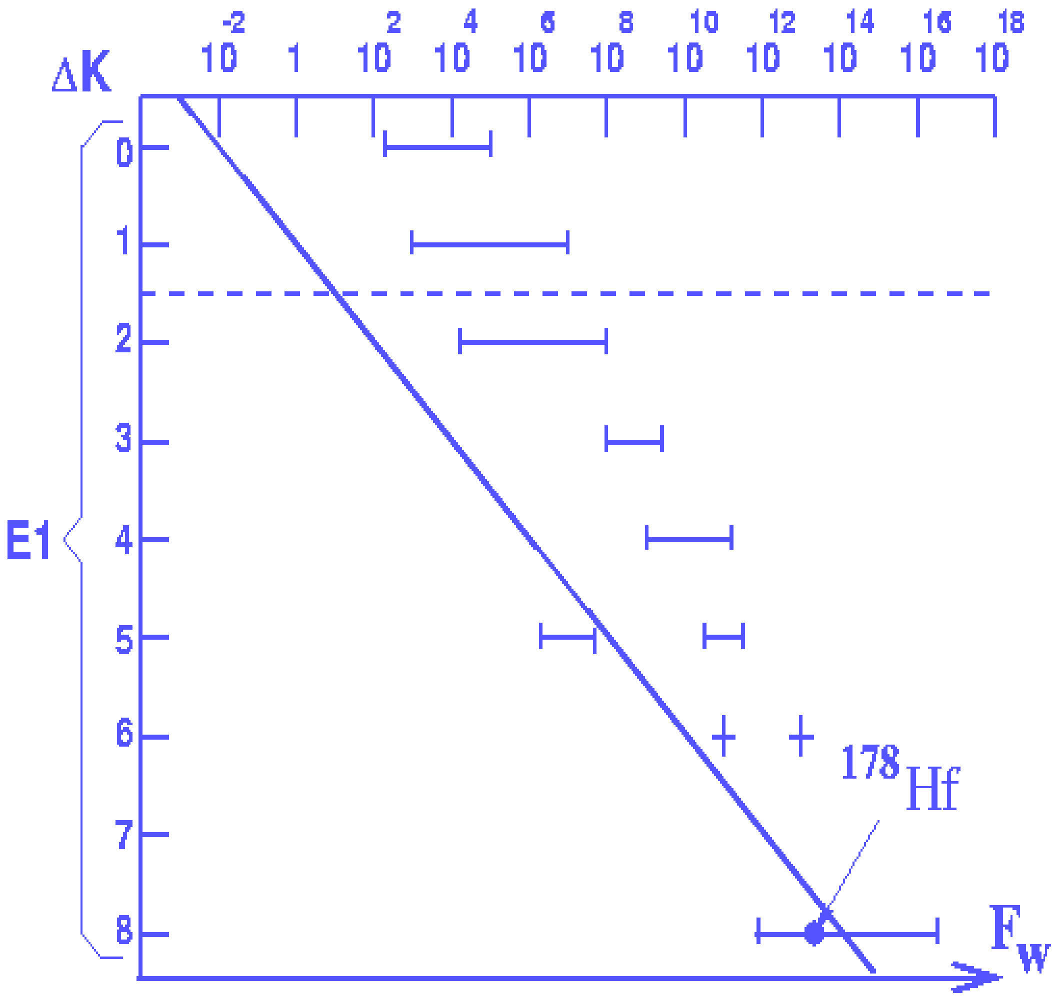
  $$f = F_W = T_{1/2}^{\gamma} / T_{1/2}^{W}$$

  where $T_{1/2}^{\gamma}$ is the partial γ-ray half-life and $T_{1/2}^{W}$ is the theoretical Weisskopf estimate

- The '<u>reduced hindrance factor</u>' is:

  $$f_\nu = f^{1/\nu} = [\ T_{1/2}^{\gamma} / T_{1/2}^{W}\ ]^{1/\nu}$$

Hindrance Factors



- The solid line shows the dependence of F_W on ΔK for some E1 transitions according to an empirical rule:

  $$\log F_W = 2\{|\Delta K| - \lambda\}$$
  $$= 2\nu$$

- i.e. F_W values increase approximately by a factor of 100 per degree of K forbiddenness

Nuclear Wobbling

- This type of rotation is predicted to only occur in triaxially deformed nuclei
- The nucleus rotates around the principal axis having the largest moment of inertia and this axis executes harmonic oscillations about the space-fixed angular momentum vector
- Its analogue in classic mechanics is an asymmetric spinning top
- Precession of the Earth

Wobbling Motion and Triaxiality

- Wobbling is a fundamental mode due to triaxiality which occurs when the axis of collective rotation does not coincide with one of the principal axes

- For a deformed rotor the Hamiltonian is:

$$H_{rot} = (\hbar^2/2\mathfrak{I}_x)\, I_x^2 + (\hbar^2/2\mathfrak{I}_y)\, I_y^2 + (\hbar^2/2\mathfrak{I}_z)\, I_z^2$$

- For a well-deformed but triaxial nucleus with $\mathfrak{I}_x \gg \mathfrak{I}_y \neq \mathfrak{I}_z$ the energy of the wobbling rotor is:

$$E(I,n_W) = (\hbar^2/2\mathfrak{I}_x)\, I(I+1) + \hbar\omega_W(I)\,(n_W + \tfrac{1}{2})$$

where n_W is the number of wobbling phonons and ω_W is the wobbling frequency

Wobbling Frequency

- The wobbling frequency is related to the rotational frequency as:

$$\boxed{\omega_W = \omega_{rot}\sqrt{[\,(\Im_x - \Im_y)(\Im_x - \Im_z) / (\Im_y \Im_z)\,]}}$$

with

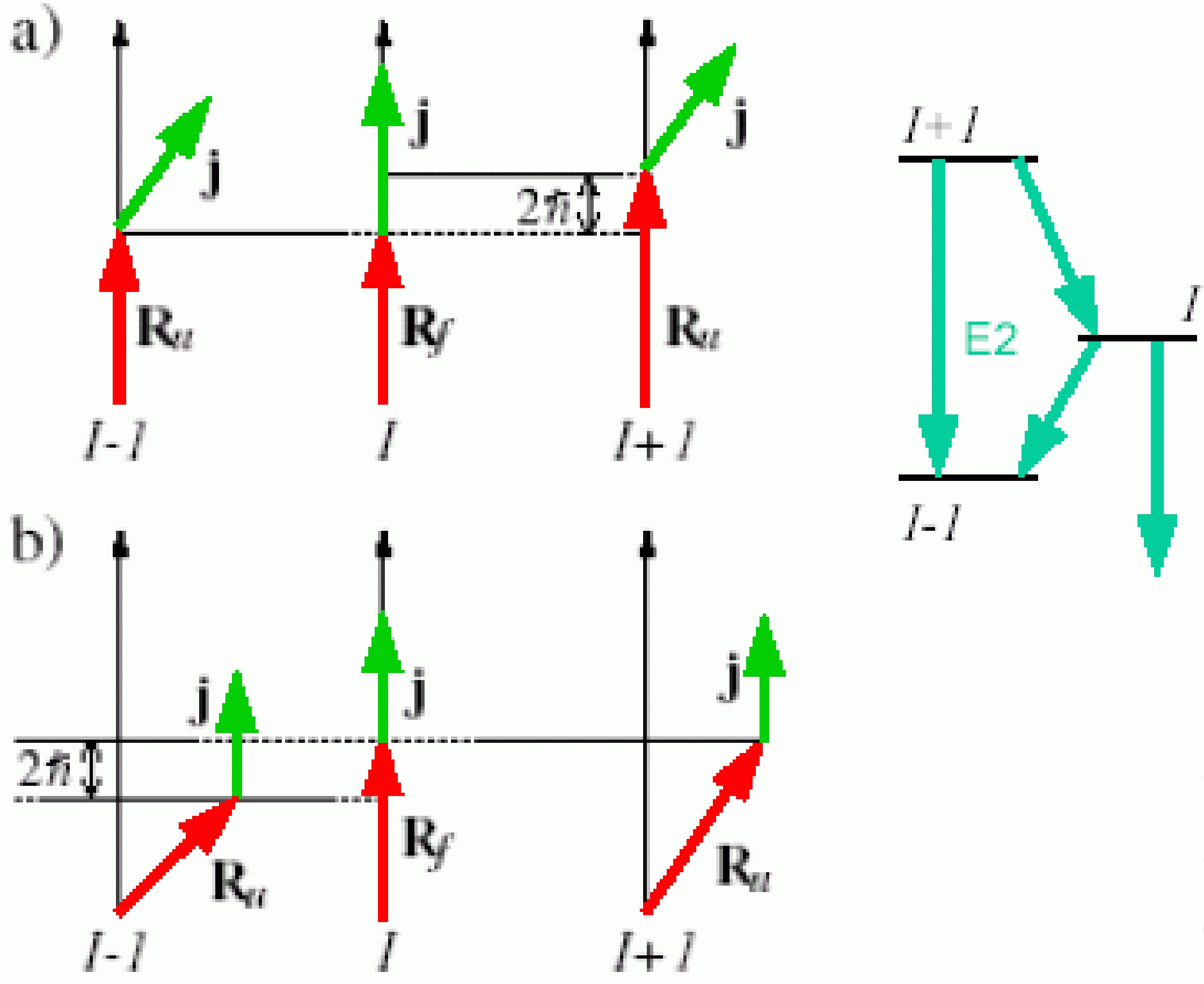
$$\boxed{\omega_{rot} = \hbar\, I / \Im_x}$$

- Note for an axially symmetric prolate nucleus, $\Im_z$ goes to zero and $\omega_W \rightarrow \infty$, i.e. there is no wobbling motion

- A family of wobbling bands is expected for $n_W = 0, 1, 2,\ldots$

Wobbling Motion

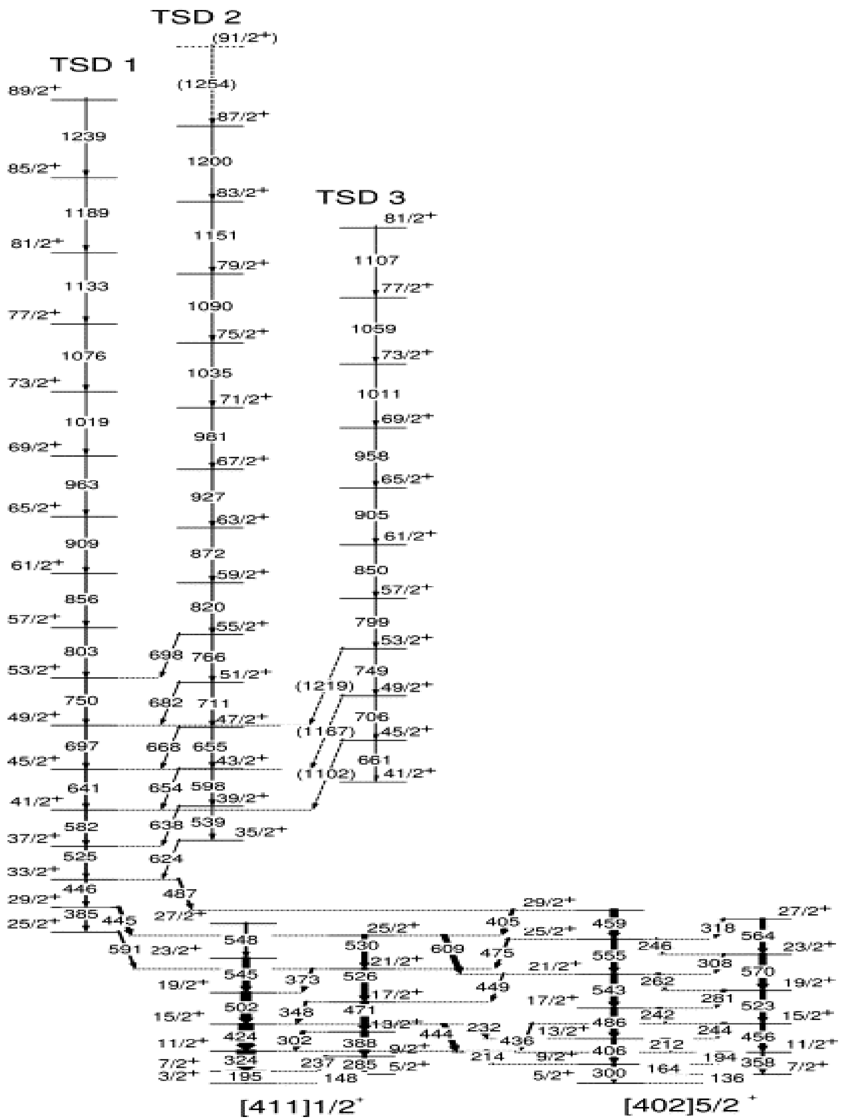
Cranking:



Wobbling:

-Requires Triaxialty!

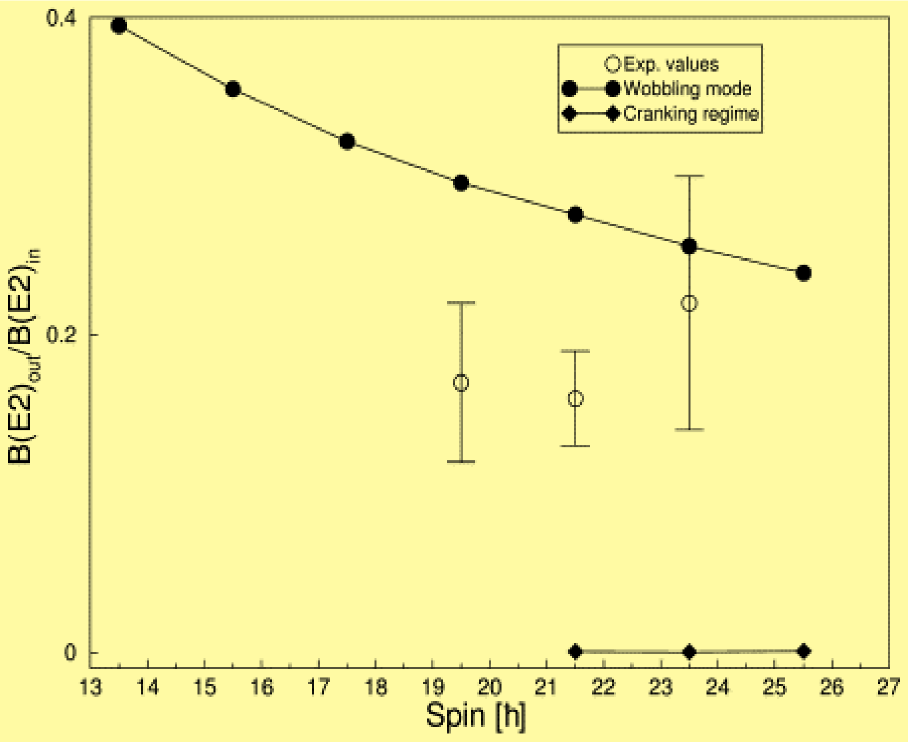
Wobbling Bands in ^{165}Lu



- *A family of wobbling bands is expected to show very similar internal structure*

- TSD (Triaxial SuperDeformed) bands 1, 2 and 3 in ^{165}Lu represent bands with 0, 1 and 2 wobbling phonons, respectively

Electromagnetic Properties



- A characteristic signature of wobbling motion is the occurrence of $\Delta I = \pm 1$ interband transitions with unusually large $B(E2)_{out}$ values that compete with the strong $\Delta I = 2$ inband transitions, $B(E2)_{in}$
- $\Delta n_W = 2$ transitions are forbidden
- Measured multipole mixing ratios for the interband $\Delta I = 1$ transitions in ^{165}Lu show them to be ~90% E2 and only ~10% M1 !

^{105}Pd: Transverse Wobbler

- Precise measurements of angular distributions and linear polarisations in ^{105}Pd have also pointed to $\Delta I = 1$ interlinking transitions with predominantly E2 (non-stretched) character

- The angular momentum of the odd neutron ($h_{11/2}$) is perpendicular to the wobbling axis

- Hence the name transverse wobbler!